8

Binomial Theorem

Short Answer Type Questions

Q. 1 Find the term independent of x, where $x \neq 0$, in the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{15}$.

Thinking Process

The general term in the expansion of $(x-a)^n$ i.e., $T_{r+1} = {}^nC_r (x)^{n-r} (-a)^r$. For the term independent of x, put $n - r = 0$, then we get the value of r.

Sol. Given expansion is $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{15}$.

Let T_{r+1} term is the general term.

Then,
$$T_{r+1} = {}^{15}C_r \left(\frac{3x^2}{2}\right)^{15-r} \left(-\frac{1}{3x}\right)^r$$
$$= {}^{15}C_r\, 3^{15-r}\, x^{30-2r}\, 2^{r-15}\, (-1)^r \cdot 3^{-r} \cdot x^{-r}$$
$$= {}^{15}C_r (-1)^r\, 3^{15-2r} 2^{r-15} x^{30-3r}$$

For independent of x,
$$30 - 3r = 0$$
$$3r = 30 \Rightarrow r = 10$$
$\because$ $T_{r+1} = T_{10+1} = 11\text{th}$ term is independent of x.

$\therefore$
$$T_{10+1} = {}^{15}C_{10} (-1)^{10}\, 3^{15-20}\, 2^{10-15}$$
$$= {}^{15}C_{10}\, 3^{-5}\, 2^{-5}$$
$$= {}^{15}C_{10} (6)^{-5}$$
$$= {}^{15}C_{10} \left(\frac{1}{6}\right)^5$$

Q. 2 If the term free from x in the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then find the value of k.

Sol. Given expansion is $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$.

Let T_{r+1} is the general term.

Then,
$$T_{r+1} = {}^{10}C_r (\sqrt{x})^{10-r} \left(\frac{-k}{x^2}\right)^r$$
$$= {}^{10}C_r (x)^{\frac{1}{2}(10-r)} (-k)^r \cdot x^{-2r}$$
$$= {}^{10}C_r x^{5-\frac{r}{2}} (-k)^r \cdot x^{-2r}$$
$$= {}^{10}C_r \, x^{5-\frac{r}{2}-2r} (-k)^r$$
$$= {}^{10}C_r \, x^{\frac{10-5r}{2}} (-k)^r$$

For free from x, $\frac{10-5r}{2} = 0$

$\Rightarrow$ $10 - 5r = 0 \Rightarrow r = 2$

Since, $T_{2+1} = T_3$ is free from x.

$\therefore$ $T_{2+1} = {}^{10}C_2 (-k)^2 = 405$

$\Rightarrow$ $\frac{10 \times 9 \times 8!}{2! \times 8!} (-k)^2 = 405$

$\Rightarrow$ $45k^2 = 405 \Rightarrow k^2 = \frac{405}{45} = 9$

$\therefore$ $k = \pm 3$

Q. 3 Find the coefficient of x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$.

Sol. Given, expansion $= (1 - 3x + 7x^2)(1 - x)^{16}$.

$= (1 - 3x + 7x^2)({}^{16}C_0 1^{16} - {}^{16}C_1 1^{15} x^1 + {}^{16}C_2 1^{14} x^2 + \ldots + {}^{16}C_{16} x^{16})$

$= (1 - 3x + 7x^2)(1 - 16x + 120x^2 + \ldots)$

$\therefore$ Coefficient of $x = -3 - 16 = -19$

Q. 4 Find the term independent of x in the expansion of $\left(3x - \frac{2}{x^2}\right)^{15}$.

Thinking Process

The general term in the expansion of $(x - a)^n$ *i.e.,* $T_{r+1} = {}^nC_r (x)^{n-r} (-a)^r$.

Sol. Given expansion is $\left(3x - \frac{2}{x^2}\right)^{15}$.

Let T_{r+1} is the general term.

$\therefore$
$$T_{r+1} = {}^{15}C_r (3x)^{15-r} \left(\frac{-2}{x^2}\right)^r = {}^{15}C_r (3x)^{15-r} (-2)^r x^{-2r}$$
$$= {}^{15}C_r 3^{15-r} x^{15-3r} (-2)^r$$

For independent of x, $15 - 3r = 0 \Rightarrow r = 5$

Since, $T_{5+1} = T_6$ is independent of x.

$$\therefore \quad T_{5+1} = {}^{15}C_5\, 3^{15-5}(-2)^5$$

$$= -\frac{15 \times 14 \times 13 \times 12 \times 11 \times 10!}{5 \times 4 \times 3 \times 2 \times 1 \times 10!} \cdot 3^{10} \cdot 2^5$$

$$= -3003 \cdot 3^{10} \cdot 2^5$$

Q. 5 Find the middle term (terms) in the expansion of

(i) $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$ (ii) $\left(3x - \frac{x^3}{6}\right)^9$

Thinking Process

In the expansion of $(a+b)^n$, if n is even, then this expansion has only one middle term i.e., $\left(\frac{n}{2}+1\right)$th term is the middle term and if n is odd, then this expansion has two middle terms i.e., $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+1}{2}+1\right)$th are two middle terms.

Sol. **(i)** Given expansion is $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$.

Here, the power of Binomial *i.e.*, $n = 10$ is even.

Since, it has one middle term $\left(\frac{10}{2}+1\right)$th term *i.e.*, 6th term.

$$\therefore \quad T_6 = T_{5+1} = {}^{10}C_5 \left(\frac{x}{a}\right)^{10-5} \left(\frac{-a}{x}\right)^5$$

$$= -{}^{10}C_5 \left(\frac{x}{a}\right)^5 \left(\frac{a}{x}\right)^5$$

$$= -\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5! \times 5 \times 4 \times 3 \times 2 \times 1} \left(\frac{x}{a}\right)^5 \left(\frac{x}{a}\right)^{-5}$$

$$= -9 \times 4 \times 7 = -252$$

(ii) Given expansion is $\left(3x - \frac{x^3}{6}\right)^9$.

Here, $n = 9$ [odd]

Since, the Binomial expansion has two middle terms *i.e.*, $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+1}{2}+1\right)$th *i.e.*, 5th term and 6th term.

$$\therefore \quad T_5 = T_{(4+1)} = {}^9C_4 (3x)^{9-4} \left(-\frac{x^3}{6}\right)^4$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!}\, 3^5\, x^5\, x^{12}\, 6^{-4}$$

$$= \frac{7 \times 6 \times 3 \times 3^1}{2^4}\, x^{17} = \frac{189}{8}\, x^{17}$$

$\therefore \quad T_6 = T_{5+1} = {}^9C_5(3x)^{9-5}\left(-\frac{x^3}{6}\right)^5$

$$= -\frac{9\times 8\times 7\times 6\times 5!}{5!\times 4\times 3\times 2\times 1}\cdot 3^4\cdot x^4\cdot x^{15}\cdot 6^{-5}$$

$$= \frac{-21\times 6}{3\times 2^5}x^{19} = \frac{-21}{16}x^{19}$$

Q. 6 Find the coefficient of x^{15} in the expansion of $(x - x^2)^{10}$.

Sol. Given expansion is $(x - x^2)^{10}$.

Let the term T_{r+1} is the general term.

$\therefore \quad T_{r+1} = {}^{10}C_r x^{10-r}(-x^2)^r$

$$= (-1)^r \cdot {}^{10}C_r \cdot x^{10-r}\cdot x^{2r}$$

$$= (-1)^r\, {}^{10}C_r\, x^{10+r}$$

For the coefficient of x^{15},

$$10 + r = 15 \Rightarrow r = 5$$

$$T_{5+1} = (-1)^5\, {}^{10}C_5\, x^{15}$$

$\therefore \quad$ Coefficient of $x^{15} = -1\,\frac{10\times 9\times 8\times 7\times 6\times 5!}{5\times 4\times 3\times 2\times 1\times 5!}$

$$= -3\times 2\times 7\times 6 = -252$$

Q. 7 Find the coefficient of $\frac{1}{x^{17}}$ in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

Thinking Process

In this type of questions, first of all find the general terms, in the expansion $(x - y)^n$ using the formula $T_{r+1} = {}^nC_r\, x^{n-r}(-y)^r$ and then put $n - r$ equal to the required power of x of which coefficient is to be find out.

Sol. Given expansion is $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

Let the term T_{r+1} contains the coefficient of $\frac{1}{x^{17}}$ *i.e.*, x^{-17}.

$\therefore \quad T_{r+1} = {}^{15}C_r (x^4)^{15-r}\left(-\frac{1}{x^3}\right)^r$

$$= {}^{15}C_r\, x^{60-4r}(-1)^r x^{-3r}$$

$$= {}^{15}C_r\, x^{60-7r}(-1)^r$$

For the coefficient x^{-17},

$$60 - 7r = -17$$

$\Rightarrow \quad 7r = 77 \Rightarrow r = 11$

$\Rightarrow \quad T_{11+1} = {}^{15}C_{11}\, x^{60-77}(-1)^{11}$

$\therefore \quad$ Coefficient of $x^{-17} = \frac{-15\times 14\times 13\times 12\times 11!}{11!\times 4\times 3\times 2\times 1}$

$$= -15\times 7\times 13 = -1365$$

Q. 8 **Find the sixth term of the expansion $(y^{1/2}+x^{1/3})^n$, if the Binomial coefficient of the third term from the end is 45.**

Sol. Given expansion is $(y^{1/2}+x^{1/3})^n$.

The sixth term of this expansion is

$$T_6 = T_{5+1} = {}^nC_5 (y^{1/2})^{n-5} (x^{1/3})^5 \quad \text{...(i)}$$

Now, given that the Binomial coefficient of the third term from the end is 45.

We know that, Binomial coefficient of third term from the end = Binomial coefficient of third term from the begining = ${}^nC_2$

$$\because \quad {}^nC_2 = 45$$

$$\Rightarrow \quad \frac{n(n-1)(n-2)!}{2!(n-2)!} = 45$$

$$\Rightarrow \quad n(n-1) = 90$$

$$\Rightarrow \quad n^2 - n - 90 = 0$$

$$\Rightarrow \quad n^2 - 10n + 9n - 90 = 0$$

$$\Rightarrow \quad n(n-10) + 9(n-10) = 0$$

$$\Rightarrow \quad (n-10)(n+9) = 0$$

$$\Rightarrow \quad (n+9) = 0 \text{ or } (n-10) = 0$$

$$\therefore \quad n = 10 \quad [\because n \neq -9]$$

From Eq. (i),

$$T_6 = {}^{10}C_5\, y^{5/2}\, x^{5/3} = 252 y^{5/2} \cdot x^{5/3}$$

Q. 9 **Find the value of r, if the coefficients of $(2r+4)$th and $(r-2)$th terms in the expansion of $(1+x)^{18}$ are equal.**

Thinking Process

Coefficient of $(r+1)$th term in the expansion of $(1+x)^n$ is ${}^nC_r$. Use this formula to solve the above problem.

Sol. Given expansion is $(1+x)^{18}$.

Now, $(2r+4)$th term *i.e.*, T_{2r+3+1}.

$$\therefore \quad T_{2r+3+1} = {}^{18}C_{2r+3}(1)^{18-2r-3}(x)^{2r+3}$$

$$= {}^{18}C_{2r+3}\, x^{2r+3}$$

Now, $(r-2)$th term *i.e.*, T_{r-3+1}.

$$\therefore \quad T_{r-3+1} = {}^{18}C_{r-3}\, x^{r-3}$$

As, $${}^{18}C_{2r+3} = {}^{18}C_{r-3} \quad [\because {}^nC_x = {}^nC_y \Rightarrow x+y=n]$$

$$\Rightarrow \quad 2r+3+r-3 = 18$$

$$\Rightarrow \quad 3r = 18$$

$$\therefore \quad r = 6$$

Q. 10 **If the coefficient of second, third and fourth terms in the expansion of $(1+x)^{2n}$ are in AP, then show that $2n^2 - 9n + 7 = 0$.**

Thinking Process

In the expansion of $(x+y)^n$, the coefficient of $(r+1)$th term is ${}^nC_r$. Use this formula to get the required coefficient. If a, b and c are in AP, then $2b = a + c$.

Sol. Given expansion is $(1+x)^{2n}$.

Now, coefficient of 2nd term $= {}^{2n}C_1$

Coefficient of 3rd term $= {}^{2n}C_2$

Coefficient of 4th term $= {}^{2n}C_3$

Given that, ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$ are in AP.

Then, $2\cdot{}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$

$$\Rightarrow \quad 2\left[\frac{2n(2n-1)(2n-2)!}{2\times 1\times(2n-2)!}\right] = \frac{2n(2n-1)!}{(2n-1)!} + \frac{2n(2n-1)(2n-2)(2n-3)!}{3!(2n-3)!}$$

$$\Rightarrow \quad n(2n-1) = n + \frac{n(2n-1)(2n-2)}{6}$$

$$\Rightarrow \quad n(12n-6) = n(6+4n^2-4n-2n+2)$$

$$\Rightarrow \quad 12n-6 = (4n^2-6n+8)$$

$$\Rightarrow \quad 6(2n-1) = 2(2n^2-3n+4)$$

$$\Rightarrow \quad 3(2n-1) = 2n^2-3n+4$$

$$\Rightarrow \quad 2n^2-3n+4-6n+3=0$$

$$\Rightarrow \quad 2n^2-9n+7=0$$

Q. 11 Find the coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^{11}$.

Sol. Given, expansion $= (1+x+x^2+x^3)^{11} = [(1+x)+x^2(1+x)]^{11}$

$= [(1+x)(1+x^2)]^{11} = (1+x)^{11}\cdot(1+x^2)^{11}$

Now, above expansion becomes

$= ({}^{11}C_0 + {}^{11}C_1x + {}^{11}C_2x^2 + {}^{11}C_3x^3 + {}^{11}C_4x^4 + \ldots)({}^{11}C_0 + {}^{11}C_1x^2 + {}^{11}C_2x^4 + \ldots)$

$= (1+11x+55x^2+165x^3+330x^4+\ldots)(1+11x^2+55x^4+\ldots)$

$\therefore$ Coefficient of $x^4 = 55 + 605 + 330 = 990$

Long Answer Type Questions

Q. 12 If p is a real number and the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, then find the value of p.

Sol. Given expansion is $\left(\frac{p}{2}+2\right)^8$.

Here, $n = 8$ [even]

Since, this Binomial expansion has only one middle term *i.e.*, $\left(\frac{8}{2}+1\right)$th = 5th term

$$T_5 = T_{4+1} = {}^8C_4\left(\frac{p}{2}\right)^{8-4}\cdot 2^4$$

$$\Rightarrow \quad 1120 = {}^8C_4\, p^4\cdot 2^{-4}\, 2^4$$

$$\Rightarrow \quad 1120 = \frac{8\times7\times6\times5\times4!}{4!\times4\times3\times2\times1}p^4$$

$$\Rightarrow \quad 1120 = 7 \times 2 \times 5 \times p^4$$

$$\Rightarrow \quad p^4 = \frac{1120}{70} = 16 \Rightarrow p^4 = 2^4$$

$$\Rightarrow \quad p^2 = 4 \Rightarrow p = \pm 2$$

Q. 13 Show that the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is $\frac{1 \times 3 \times 5 \times \ldots \times (2n-1)}{n!} \times (-2)^n$.

Sol. Given, expansion is $\left(x - \frac{1}{x}\right)^{2n}$. This Binomial expansion has even power. So, this has one middle term.

i.e., $\left(\frac{2n}{2} + 1\right)$th term $= (n+1)$th term

$$T_{n+1} = {}^{2n}C_n (x)^{2n-n} \left(-\frac{1}{x}\right)^n = {}^{2n}C_n \, x^n (-1)^n x^{-n}$$

$$= {}^{2n}C_n (-1)^n = (-1)^n \frac{(2n)!}{n!n!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \ldots (2n-1)(2n)}{n!n!}(-1)^n$$

$$= \frac{1 \cdot 3 \cdot 5 \ldots (2n-1) \cdot 2 \cdot 4 \cdot 6 \ldots (2n)}{12 \cdot 3 \cdot \ldots n(n!)}(-1)^n$$

$$= \frac{1 \cdot 3 \cdot 5 \ldots (2n-1) \,.\, 2^n (1 \cdot 2 \cdot 3 \ldots n)(-1)^n}{(1 \cdot 2 \cdot 3 \ldots n)(n!)}$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (2n-1)]}{n!}(-2)^n$$

Hence proved.

Q. 14 Find n in the Binomial $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, if the ratio of 7th term from the beginning to the 7th term from the end is $\frac{1}{6}$.

Sol. Here, the Binomial expansion is $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$.

Now, 7th term from beginning $T_7 = T_{6+1} = {}^nC_6 (\sqrt[3]{2})^{n-6} \left(\frac{1}{\sqrt[3]{3}}\right)^6$ ...(i)

and 7th term from end *i.e.,* T_7 from the beginning of $\left(\frac{1}{\sqrt[3]{3}} + \sqrt[3]{2}\right)^n$

i.e., $$T_7 = {}^nC_6 \left(\frac{1}{\sqrt[3]{3}}\right)^{n-6} (\sqrt[3]{2})^6 \qquad \text{...(ii)}$$

Given that, $$\frac{{}^nC_6 (\sqrt[3]{2})^{n-6} \left(\frac{1}{\sqrt[3]{3}}\right)^6}{{}^nC_6 \left(\frac{1}{\sqrt[3]{3}}\right)^{n-6} (\sqrt[3]{2})^6} = \frac{1}{6} \Rightarrow \frac{2^{\frac{n-6}{3}} \cdot 3^{-6/3}}{3^{-\left(\frac{n-6}{3}\right)} \cdot 2^{6/3}} = \frac{1}{6}$$

$$\Rightarrow \left(2^{\frac{n-6}{3}} \cdot 2^{\frac{-6}{3}}\right)\left(3^{\frac{-6}{3}} \cdot 3^{\frac{(n-6)}{3}}\right) = 6^{-1}$$

$$\Rightarrow \quad \left(2^{\frac{n-6}{3}-\frac{6}{3}}\right)\cdot\left(3^{\frac{n-6}{3}-\frac{6}{3}}\right) = 6^{-1} \Rightarrow (2\cdot 3)^{\frac{n}{3}-4} = 6^{-1}$$

$$\Rightarrow \quad \frac{n}{3} - 4 = -1 \Rightarrow \frac{n}{3} = 3$$

$$\therefore \quad n = 9$$

Q. 15 In the expansion of $(x+a)^n$, if the sum of odd terms is denoted by O and the sum of even term by E. Then, prove that

(i) $O^2 - E^2 = (x^2 - a^2)^n$.

(ii) $4OE = (x+a)^{2n} - (x-a)^{2n}$.

Sol. **(i)** Given expansion is $(x+a)^n$.

$\therefore \quad (x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 + {}^nC_3 x^{n-3} a^3 + \ldots + {}^nC_n a^n$

Now, sum of odd terms

i.e., $\quad O = {}^nC_0 x^n + {}^nC_2 x^{n-2} a^2 + \ldots$

and sum of even terms

i.e., $\quad E = {}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \ldots$

$\because \quad (x+a)^n = O + E \quad$ ...(i)

Similarly, $\quad (x-a)^n = O - E \quad$ ...(ii)

$\therefore \quad (O+E)(O-E) = (x+a)^n (x-a)^n \quad$ [on multiplying Eqs. (i) and (ii)]

$\Rightarrow \quad O^2 - E^2 = (x^2 - a^2)^n$

(ii) $4\,OE = (O+E)^2 - (O-E)^2 = [(x+a)^n]^2 - [(x-a)^n]^2 \quad$ [from Eqs. (i) and (ii)]

$= (x+a)^{2n} - (x-a)^{2n}$ **Hence proved**.

Q. 16 If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, then prove that its coefficient is $\dfrac{2n!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}$.

Sol. Given expansion is $\left(x^2 + \frac{1}{x}\right)^{2n}$.

Let x^p occur in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$.

$$T_{r+1} = {}^{2n}C_r (x^2)^{2n-r}\left(\frac{1}{x}\right)^r$$

$$= {}^{2n}C_r x^{4n-2r} x^{-r} = {}^{2n}C_r x^{4n-3r}$$

Let $\quad 4n - 3r = p$

$\Rightarrow \quad 3r = 4n - p \Rightarrow r = \dfrac{4n-p}{3}$

$$\therefore \quad \text{Coefficient of } x^p = {}^{2n}C_r = \frac{(2n)!}{r!\,(2n-r)!} = \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(2n - \frac{4n-p}{3}\right)!}$$

$$= \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{6n-4n+p}{3}\right)!} = \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}$$

Q. 17 Find the term independent of x in the expansion of

$$(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9.$$

Sol. Given expansion is $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

Now, consider $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

$$T_{r+1} = {}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$$

$$= {}^9C_r\left(\frac{3}{2}\right)^{9-r} x^{18-2r}\left(-\frac{1}{3}\right)^r x^{-r} = {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r}$$

Hence, the general term in the expansion of $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

$$= {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r} + {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{19-3r} + 2\cdot{}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{21-3r}$$

For term independent of x, putting $18-3r=0$, $19-3r=0$ and $21-3r=0$, we get

$$r=6,\ r=19/3,\ r=7$$

Since, the possible value of r are 6 and 7.

Hence, second term is not independent of x.

$\therefore$ The term independent of x is ${}^9C_6\frac{3^{9-6}}{2}\left(-\frac{1}{3}\right)^6 + 2\cdot{}^9C_7\frac{3^{9-7}}{2}\left(-\frac{1}{3}\right)^7$

$$= \frac{9\times8\times7\times6!}{6!\times3\times2}\cdot\frac{3^3}{2^3}\cdot\frac{1}{3^6} - 2\cdot\frac{9\times8\times7!}{7!\times2\times1}\cdot\frac{3^2}{2^2}\cdot\frac{1}{3^7}$$

$$= \frac{84}{8}\cdot\frac{1}{3^3} - \frac{36}{4}\cdot\frac{2}{3^5} = \frac{7}{18} - \frac{2}{27} = \frac{21-4}{54} = \frac{17}{54}$$

Objective Type Questions

Q. 18 The total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplification is

(a) 50 (b) 202 (c) 51 (d) None of these

Sol. *(c)* Here, $(x+a)^{100}+(x-a)^{100}$

Total number of terms is 102 in the expansion of $(x+a)^{100}+(x-a)^{100}$

50 terms of $(x+a)^{100}$ cancel out 50 terms of $(x-a)^{100}$. 51 terms of $(x+a)^{100}$ get added to the 51 terms of $(x-a)^{100}$.

Alternate Method

$$(x+a)^{100}+(x-a)^{100} = {}^{100}C_0\,x^{100} + {}^{100}C_1\,x^{99}a + \ldots + {}^{100}C_{100}\,a^{100}$$
$$+{}^{100}C_0\,x^{100} - {}^{100}C_1\,x^{99}a + \ldots + {}^{100}C_{100}\,a^{100}$$
$$= 2\underbrace{[{}^{100}C_0\,x^{100} + {}^{100}C_2\,x^{98}a^2 + \ldots + {}^{100}C_{100}\,a^{100}]}_{51 \text{ terms}}$$

Q. 19 If the integers $r > 1$, $n > 2$ and coefficients of $(3r)$th and $(r + 2)$nd terms in the Binomial expansion of $(1 + x)^{2n}$ are equal, then

(a) $n = 2r$ (b) $n = 3r$

(c) $n = 2r + 1$ (d) None of these

Thinking Process

In the expansion of $(x + y)^n$, the coefficient of $(r + 1)$th term is ${}^nC_r$.

Sol. *(a)* Given that, $r > 1$, $n > 2$ and the coefficients of $(3r)$th and $(r + 2)$th term are equal in the expansion of $(1 + x)^{2n}$.

Then, $T_{3r} = T_{3r-1+1} = {}^{2n}C_{3r-1}\, x^{3r-1}$

and $T_{r+2} = T_{r+1+1} = {}^{2n}C_{r+1}\, x^{r+1}$

Given, ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$ $[\because {}^nC_x = {}^nC_y \Rightarrow x + y = n]$

$\Rightarrow$ $3r - 1 + r + 1 = 2n$

$\Rightarrow$ $4r = 2n \Rightarrow n = \dfrac{4r}{2}$

$\therefore$ $n = 2r$

Q. 20 The two successive terms in the expansion of $(1 + x)^{24}$ whose coefficients are in the ratio 1 : 4 are

(a) 3rd and 4th (b) 4th and 5th

(c) 5th and 6th (d) 6th and 7th

Sol. *(c)* Let two successive terms in the expansion of $(1 + x)^{24}$ are $(r + 1)$th and $(r + 2)$th terms.

$\therefore$ $T_{r+1} = {}^{24}C_r\, x^r$

and $T_{r+2} = {}^{24}C_{r+1}\, x^{r+1}$

Given that, $\dfrac{{}^{24}C_r}{{}^{24}C_{r+1}} = \dfrac{1}{4}$

$$\Rightarrow \frac{\dfrac{(24)!}{r!(24-r)!}}{\dfrac{(24)!}{(r+1)!(24-r-1)!}} = \frac{1}{4}$$

$$\Rightarrow \frac{(r+1)\, r!\, (23-r)!}{r!(24-r)\,(23-r)!} = \frac{1}{4}$$

$$\Rightarrow \frac{r+1}{24-r} = \frac{1}{4} \Rightarrow 4r + 4 = 24 - r$$

$\Rightarrow$ $5r = 20 \Rightarrow r = 4$

$\therefore$ $T_{4+1} = T_5$ and $T_{4+2} = T_6$

Hence, 5th and 6th terms.

Q. 21 The coefficient of x^n in the expansion of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ are in the ratio

(a) 1 : 2 (b) 1 : 3

(c) 3 : 1 (d) 2 : 1

Sol. *(d)* $\because$ Coefficient of x^n in the expansion of $(1+x)^{2n} = {}^{2n}C_n$

and coefficient of x^n in the expansion of $(1+x)^{2n-1} = {}^{2n-1}C_n$

$$\because \quad \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{\frac{(2n)!}{n!n!}}{\frac{(2n-1)!}{n!(n-1)!}}$$

$$= \frac{(2n)!\,n!\,(n-1)!}{n!\,n!\,(2n-1)!}$$

$$= \frac{2n\,(2n-1)!\,n!\,(n-1)!}{n!\,n(n-1)!\,(2n-1)!}$$

$$= \frac{2n}{n} = \frac{2}{1} = 2:1$$

Q. 22 If the coefficients of 2nd, 3rd and the 4th terms in the expansion of $(1+x)^n$ are in AP, then the value of n is

(a) 2 (b) 7

(c) 11 (d) 14

Sol. *(b)* The expansion of $(1+x)^n$ is ${}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_nx^n$

$\therefore$ Coefficient of 2nd term $= {}^nC_1$,

Coefficient of 3rd term $= {}^nC_2$,

and coefficient of 4th term $= {}^nC_3$.

Given that, ${}^nC_1$, ${}^nC_2$ and ${}^nC_3$ are in AP.

$$\therefore \quad 2\,{}^nC_2 = {}^nC_1 + {}^nC_3$$

$$\Rightarrow \quad 2\left[\frac{(n)!}{(n-2)!\,2!}\right] = \frac{(n)!}{(n-1)!} + \frac{(n)!}{3!\,(n-3)!}$$

$$\Rightarrow \quad \frac{2\cdot n\,(n-1)\,(n-2)!}{(n-2)!\,2!} = \frac{n\,(n-1)!}{(n-1)!} + \frac{n(n-1)\,(n-2)\,(n-3)!}{3\cdot 2\cdot 1\,(n-3)!}$$

$$\Rightarrow \quad n(n-1) = n + \frac{n(n-1)\,(n-2)}{6}$$

$$\Rightarrow \quad 6n - 6 = 6 + n^2 - 3n + 2$$

$$\Rightarrow \quad n^2 - 9n + 14 = 0$$

$$\Rightarrow \quad n^2 - 7n - 2n + 14 = 0$$

$$\Rightarrow \quad n(n-7) - 2\,(n-7) = 0$$

$$\Rightarrow \quad (n-7)\,(n-2) = 0$$

$$\therefore \quad n = 2 \text{ or } n = 7$$

Since, $n = 2$ is not possible.

$$\therefore \quad n = 7$$

Q. 23 If A and B are coefficient of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then $\frac{A}{B}$ equals to

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) $\frac{1}{n}$

Sol. *(b)* Since, the coefficient of x^n in the expansion of $(1+x)^{2n}$ is ${}^{2n}C_n$.

$$\therefore \quad A = {}^{2n}C_n$$

Now, the coefficient of x^n in the expansion of $(1+x)^{2n-1}$ is ${}^{2n-1}C_n$.

$$\therefore \quad B = {}^{2n-1}C_n$$

Now, $$\frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{2}{1} = 2$$

Same as solution No. 21.

Q. 24 If the middle term of $\left(\frac{1}{x} + x\sin x\right)^{10}$ is equal to $7\frac{7}{8}$, then the value of x is

(a) $2n\pi + \frac{\pi}{6}$ (b) $n\pi + \frac{\pi}{6}$

(c) $n\pi + (-1)^n \frac{\pi}{6}$ (d) $n\pi + (-1)^n \frac{\pi}{3}$

Sol. *(c)* Given expansion is $\left(\frac{1}{x} + x\sin x\right)^{10}$.

Since, $n = 10$ is even, so this expansion has only one middle term *i.e.,* 6th term.

$$\therefore \quad T_6 = T_{5+1} = {}^{10}C_5 \left(\frac{1}{x}\right)^{10-5} (x\sin x)^5$$

$$\Rightarrow \quad \frac{63}{8} = {}^{10}C_5\, x^{-5} x^5 \sin^5 x$$

$$\Rightarrow \quad \frac{63}{8} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5!}{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 5!}\sin^5 x$$

$$\Rightarrow \quad \frac{63}{8} = 2\cdot 9\cdot 2\cdot 7\cdot \sin^5 x$$

$$\Rightarrow \quad \sin^5 x = \frac{1}{32}$$

$$\Rightarrow \quad \sin^5 x = \left(\frac{1}{2}\right)^5$$

$$\Rightarrow \quad \sin x = \frac{1}{2}$$

$$\therefore \quad x = n\pi + (-1)^n \, \pi/6$$

Fillers

Q. 25 The largest coefficient in the expansion of $(1+x)^{30}$ is ....

Thinking Process

In the expansion of $(1+x)^n$, the largest coefficient is ${}^nC_{n/2}$ (when n is even).

Sol. Largest coefficient in the expansion of $(1+x)^{30} = {}^{30}C_{30/2} = {}^{30}C_{15}$

Q. 26 The number of terms in the expansion of $(x+y+z)^n$ ....

Sol. Given expansion is $(x+y+z)^n = [x+(y+z)]^n$.

$$[x+(y+z)]^n = {}^nC_0x^n + {}^nC_1x^{n-1}(y+z) + {}^nC_2x^{n-2}(y+z)^2 + \ldots + {}^nC_n(y+z)^n$$

$\therefore$ Number of terms $= 1+2+3+\ldots+n+(n+1)$

$$= \frac{(n+1)(n+2)}{2}$$

Q. 27 In the expansion of $\left(x^2 - \frac{1}{x^2}\right)^{16}$, the value of constant term is ....

Sol. Let constant be T_{r+1}.

$$\therefore \quad T_{r+1} = {}^{16}C_r(x^2)^{16-r}\left(-\frac{1}{x^2}\right)^r$$

$$= {}^{16}C_r\, x^{32-2r}(-1)^r x^{-2r}$$

$$= {}^{16}C_r\, x^{32-4r}(-1)^r$$

For constant term, $32 - 4r = 0 \Rightarrow r = 8$

$$\therefore \quad T_{8+1} = {}^{16}C_8$$

Q. 28 If the seventh term from the beginning and the end in the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ are equal, then n equals to ....

Sol. Given expansions is $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$.

$$\therefore \quad T_7 = T_{6+1} = {}^nC_6(\sqrt[3]{2})^{n-6}\left(\frac{1}{\sqrt[3]{3}}\right)^6 \quad \ldots(i)$$

Since, T_7 from end is same as the T_7 from beginning of $\left(\frac{1}{\sqrt[3]{3}} + \sqrt[3]{2}\right)^n$.

Then, $$T_7 = {}^nC_6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6}(\sqrt[3]{2})^6 \quad \ldots(ii)$$

Given that, $${}^nC_6(2)^{\frac{n-6}{3}}(3)^{-6/3} = {}^nC_6(3)^{-\frac{(n-6)}{3}}2^{6/3}$$

$$\Rightarrow \quad (2)^{\frac{n-12}{3}} = \left(\frac{1}{3^{1/3}}\right)^{n-12}$$

which is true, when $\frac{n-12}{3} = 0$.

$$\Rightarrow \quad n - 12 = 0 \Rightarrow n = 12$$

Q. 29 The coefficient of $a^{-6}b^4$ in the expansion of $\left(\frac{1}{a}-\frac{2b}{3}\right)^{10}$ is ....

Thinking Process

In the expansion of $(x-a)^n$, $T_{r+1} = {}^nC_r\, x^{n-r}(-a)^r$

Sol. Given expansion is $\left(\frac{1}{a}-\frac{2b}{3}\right)^{10}$.

Let T_{r+1} has the coefficient of $a^{-6}b^4$.

$$\therefore \quad T_{r+1} = {}^{10}C_r\left(\frac{1}{a}\right)^{10-r}\left(-\frac{2b}{3}\right)^r$$

For coefficient of $a^{-6}b^4$, $\quad 10-r=6 \Rightarrow r=4$

$$\text{Coefficient of } a^{-6}b^4 = {}^{10}C_4(-2/3)^4$$

$$\therefore \quad = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6!}{6!\cdot 4\cdot 3\cdot 2\cdot 1}\cdot\frac{2^4}{3^4} = \frac{1120}{27}$$

Q. 30 Middle term in the expansion of $(a^3+ba)^{28}$ is ....

Sol. Given expansion is $(a^3+ba)^{28}$.

$\because \quad n = 28 \quad$ [even]

$\therefore \quad$ Middle term $= \left(\frac{28}{2}+1\right)$th term $=$ 15th term

$$\therefore \quad T_{15} = T_{14+1}$$
$$= {}^{28}C_{14}(a^3)^{28-14}(ba)^{14}$$
$$= {}^{28}C_{14}\, a^{42}b^{14}a^{14}$$
$$= {}^{28}C_{14}\, a^{56}b^{14}$$

Q. 31 The ratio of the coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ is ....

Sol. Given expansion is $(1+x)^{p+q}$.

$\therefore \quad$ Coefficient of $x^p = {}^{p+q}C_p$

and $\quad$ coefficient of $x^q = {}^{p+q}C_q$

$$\therefore \quad \frac{{}^{p+q}C_p}{{}^{p+q}C_q} = \frac{{}^{p+q}C_p}{{}^{p+q}C_p} = 1:1$$

Q. 32 The position of the term independent of x in the expansion of $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$ is ....

Sol. Given expansion is $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$.

Let the constant term be T_{r+1}.

Then, $T_{r+1} = {}^{10}C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \left(\frac{3}{2x^2}\right)^r$

$$= {}^{10}C_r \cdot x^{\frac{10-r}{2}} \cdot 3^{\frac{-10+r}{2}} \cdot 3^r \cdot 2^{-r} \cdot x^{-2r}$$

$$= {}^{10}C_r \; x^{\frac{10-5r}{2}} \; 3^{\frac{-10+3r}{2}} \; 2^{-r}$$

For constant term, $10 - 5r = 0 \Rightarrow r = 2$

Hence, third term is independent of x.

Q. 33 If 25^{15} is divided by 13, then the remainder is ....

Sol. Let $25^{15} = (26-1)^{15}$

$$= {}^{15}C_0\, 26^{15} - {}^{15}C_1 26^{14} + \ldots - {}^{15}C_{15}$$

$$= {}^{15}C_0 26^{15} - {}^{15}C_1 26^{14} + \ldots - 1 - 13 + 13$$

$$= {}^{15}C_0\, 26^{15} - {}^{15}C_1\, 26^{14} + \ldots - 13 + 12$$

It is clear that, when 25^{15} is divided by 13, then remainder will be 12.

True/False

Q. 34 The sum of the series $\sum_{r=0}^{10} {}^{20}C_r$ is $2^{19} + \frac{{}^{20}C_{10}}{2}$.

Sol. *False*

Given series $= \sum_{r=0}^{10} {}^{20}C_r = {}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \ldots + {}^{20}C_{10}$

$$= {}^{20}C_0 + {}^{20}C_1 + \ldots + {}^{20}C_{10} + {}^{20}C_{11} + \ldots {}^{20}C_{20} - ({}^{20}C_{11} + \ldots + {}^{20}C_{20})$$

$$= 2^{20} - ({}^{20}C_{11} + \ldots + {}^{20}C_{20})$$

Hence, the given statement is false.

Q. 35 The expression $7^9 + 9^7$ is divisible by 64.

Sol. *True*

Given expression $= 7^9 + 9^7 = (1+8)^7 - (1-8)^9$

$$= ({}^7C_0 + {}^7C_1 8 + {}^7C_2 8^2 + \ldots + {}^7C_7 8^7) - ({}^9C_0 - {}^9C_1 8 + {}^9C_2 8^2 \ldots - {}^9C_9 8^9)$$

$$= (1 + 7 \times 8 + 21 \times 8^2 + \ldots) - (1 - 9 \times 8 + 36 \times 8^2 + \ldots - 8^9)$$

$$= (7 \times 8 + 9 \times 8) + (21 \times 8^2 - 36 \times 8^2) + \ldots$$

$$= 2 \times 64 + (21 - 36)64 + \ldots$$

which is divisible by 64.

Hence, the statement is true.

Q. 36 The number of terms in the expansion of $[(2x + y^3)^4]^7$ is 8.

Sol. *False*

Given expansion is $[(2x + y^3)^4]^7 = (2x + y^3)^{28}$.

Since, this expansion has 29 terms.

So, the given statement is false.

Q. 37 The sum of coefficients of the two middle terms in the expansion of $(1+x)^{2n-1}$ is equal to ${}^{2n-1}C_n$.

Sol. ***False***

Here, the Binomial expansion is $(1+x)^{2n-1}$.

Since, this expansion has two middle term *i.e.*, $\left(\frac{2n-1+1}{2}\right)$th term and $\left(\frac{2n-1+1}{2}+1\right)$th term *i.e.*, nth term and $(n+1)$th term.

$$\therefore \quad \text{Coefficient of } n\text{th term} = {}^{2n-1}C_{n-1}$$

$$\text{Coefficient of } (n+1)\text{th term} = {}^{2n-1}C_n$$

$$\text{Sum of coefficients} = {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$$

$$= {}^{2n-1+1}C_n = {}^{2n}C_n \qquad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$$

Q. 38 The last two digits of the numbers 3^{400} are 01.

Sol. ***True***

Given that, $3^{400} = 9^{200} = (10-1)^{200}$

$$\Rightarrow \quad (10-1)^{200} = {}^{200}C_0 10^{200} - {}^{200}C_1 10^{199} + \dots - {}^{200}C_{199} 10^1 + {}^{200}C_{200}\, 1^{200}$$

$$\Rightarrow \quad (10-1)^{200} = 10^{200} - 200 \times 10^{199} + \dots - 10 \times 200 + 1$$

So, it is clear that the last two digits are 01.

Q. 39 If the expansion of $\left(x - \frac{1}{x^2}\right)^{2n}$ contains a term independent of x, then n is a multiple of 2.

Sol. ***False***

Given Binomial expansion is $\left(x - \frac{1}{x^2}\right)^{2n}$.

Let T_{r+1} term is independent of x.

Then,
$$T_{r+1} = {}^{2n}C_r (x)^{2n-r} \left(-\frac{1}{x^2}\right)^r$$

$$= {}^{2n}C_r\, x^{2n-r} (-1)^r x^{-2r} = {}^{2n}C_r\, x^{2n-3r} (-1)^r$$

For independent of x,

$$2n - 3r = 0$$

$$\therefore \quad r = \frac{2n}{3},$$

which is not a integer.

So, the given expansion is not possible.

Q. 40 The number of terms in the expansion of $(a+b)^n$, where $n \in N$, is one less than the power n.

Sol. ***False***

We know that, the number of terms in the expansion of $(a+b)^n$, where $n \in N$, is one more than the power n.